

1st order PDE

① Method of characteristic (Lagrange Method)

$$a(x, y, u) u_x + b(x, y, u) u_y - c(x, y, u) = 0$$

$$\frac{dx}{a(x, y, u)} = \frac{dy}{b(x, y, u)} = \frac{du}{c(x, y, u)}$$

$$f(\phi, \psi) = 0 \rightarrow u(x, y) - u = 0$$

② Canonical Method [Cancelled] ممكن
second ممكن

③ Method of separation of Variable

$$f(x, y, u, u_x, u_y) = 0$$

فصل المتغيرات

$$U = f(x) \cdot g(y) \Rightarrow u_x = f'g \rightarrow u_y = fg'$$

$$U = f(x) + g(y) \Rightarrow u_x = f' \rightarrow u_y = g'$$

Example 2.7.1

solve the IVP

$$u_x + 2u_y = 0$$

$$u(0, y) = 4e^{-2y}$$

$$u = f(x) \cdot g(y)$$

$$u_x = \frac{df}{dx} \cdot g, \quad u_y = \frac{dg}{dy} \cdot f$$

$$\frac{df}{dx} \cdot g + 2f \frac{dg}{dy} = 0$$

Divide by $f \cdot g$

$$\frac{\frac{df}{dx} \cdot g}{f \cdot g} + 2 \frac{f \frac{dg}{dy}}{f \cdot g} = 0$$

$$\frac{\frac{df}{dx}}{f} + 2 \frac{\frac{dg}{dy}}{g} = 0$$

$$\frac{df/dx}{f} = -2 \frac{dg/dy}{g} = \text{Constant}$$

$$\frac{f'}{f} = 1$$

$$\frac{f'}{f} = 1 \rightarrow \int \frac{f'}{f} = \int 1 dx$$

$$\therefore \ln f = 1x + \text{Constant}$$

$$\therefore \{f = C_1 e^{\lambda x}\}$$

$$-2 \frac{dg/dy}{g} = 1$$

$$\int \frac{dg}{g} = \int \frac{-1}{2} dy$$

$$\therefore \ln g = \frac{-1}{2} y + \text{Constant}$$

$$g(y) = e^{\frac{-1}{2} y} C_2$$

$$g(y) = C_2 e^{\frac{-1}{2} y}$$

$$u(x, y) = f(x) \cdot g(y) \\ = C_1 e^{\lambda x} * C_2 e^{\frac{-1}{2} y}$$

$$u(x, y) = C e^{\lambda x - \frac{1}{2} y}$$

$$u(x, y) = C e^{\lambda(x - \frac{y}{2})} \quad \rightarrow \text{General solution}$$

$$u(0, y) = 4 e^{-2y} = C e^{\lambda(0 - \frac{y}{2})}$$

$$u(0, y) = 4 e^{-2y} = C e^{\frac{-1}{2} y}$$

$$C = 4, \lambda = 4$$

$$\therefore u(x, y) = 4 e^{u(x - \frac{y}{2})}$$

#

2.1

$$u(x, y) = f(x) + g(y)$$

$$u_x = \frac{df}{dx}, u_y = \frac{dg}{dy}$$

$$\frac{df}{dx} + 2 \frac{dg}{dy} = 0$$

$$\frac{df}{dx} = -2 \frac{dg}{dy} = 1$$

$$f = \lambda x + C_1, g = \frac{1}{2} y + C_2$$

$$u(x, y) = \lambda x - \frac{1}{2} y + C_1 + C_2$$

$$u(x, y) = \lambda x - \frac{1}{2} y + C$$

To find λ and C

$$u(0,y) = 4e^{-\frac{y}{2}}$$

$$u(0,y) = -\frac{\lambda}{2}y + C$$

$$4e^{-\frac{y}{2}} = -\frac{\lambda}{2}y + C$$

λ and C Cannot be got.

$u(x,y) = f(x) + g(y)$ is not suitable for their initial condition - switch

ex solve $y^2 u_x^2 + x u_y = (xyu)^2$

$$u = f(x) \cdot g(y)$$

$$u_x = f' \cdot g, \quad u_y = f \cdot g'$$

$$y^2 (f')^2 \cdot g^2 + x \cdot f \cdot g' = (xyfg)^2$$

Divide by $(xyfg)^2$

$$\frac{y^2 (f')^2 \cdot g^2}{(xyfg)^2} + \frac{xfg'}{(xyfg)^2} = 1$$

$$= \frac{(f')^2}{(xf)^2} + \frac{g'}{xfy^2g^2} = 1$$

$$\frac{f'^2}{x^2 f^2} + \frac{g'}{xfy^2g^2} = 1$$

Multiply by $x \cdot f(x)$

$$\frac{f'^2}{xf} + \frac{g'}{y^2g^2} = x \cdot f$$

$$\frac{f'^2}{xf} - xf + \frac{g'}{y^2g^2} = 0$$

$$\frac{f'^2}{xf} - xf = -\frac{g'}{y^2g^2} = \lambda$$

$$f(x) \rightarrow \frac{f'^2}{xf} - xf = \lambda, \quad g(y) \rightarrow \frac{-g'}{y^2g^2} = \lambda$$

$$f'^2 - x^2 f^2 = 1 \quad x f$$

$$f'^2 - x^2 f^2 - 1 \quad x f = 0$$

Put $f' = P$

$$P^2 - x^2 f^2 - 1 \quad x f = 0$$

$$P^2 - x f [x f + 1] = 0$$

$$(P - \sqrt{x f (x f + 1)}) (P + \sqrt{x f (x f + 1)}) = 0$$

$$\Rightarrow \frac{-g'}{y^2 g^2} = 1$$

$$\int \frac{-dg}{g^2} = \int y^2 1 dy$$

$$\frac{1}{g} = \frac{1}{3} y^3 + C_2$$

$$g = \frac{C_2}{\frac{1}{3} y^3}$$

* Can we use the sum.

$$u(x, y) = f(x) + g(y)$$

$$u_x = f', \quad u_y = g'$$

$$y^2 u_x^2 + x u_y^2 = (x y u)^2$$

$$y^2 f'^2 + x g'^2 = [x y (f + g)]^2$$

$$y^2 f'^2 + x g'^2 = x^2 y^2 (f + g)^2$$

Sum is not useful #

WIP

solve

$$u(x+y) u_x + u(x-y) u_y = x^2 + y^2$$

$$u(x+y) u_x + u(x-y) u_y - (x^2 + y^2) = 0$$

$$a = u(x+y)$$

$$b = u(x-y)$$

$$-C = -(x^2 + y^2)$$

$$\frac{dx}{a} = \frac{dy}{b} = \frac{du}{c}$$

$$\frac{\frac{dx}{a}}{u(x+y)} = \frac{\frac{dy}{b}}{u(x-y)} = \frac{\frac{du}{c}}{x^2+y^2}$$

Note

$$\frac{a}{b} = \frac{c}{d} = \frac{a+c}{b+d} = \frac{a-c}{b-d}$$

$$= \frac{xa+yc}{xb+yd} = \frac{xa-yc}{xb-yd}$$

$$\frac{-x dx}{-xu(x+y)} = \frac{y dy}{uy(x-y)} = \frac{+u dy}{+u(x^2+y^2)}$$

$$= \frac{-x dx + y dy + u dy}{-x^2 u + xyu + xyu - uy^2 + ux^2 + uy^2}$$

$$= \frac{-x dx + y dy + u dy}{-x^2 u + xyu + xyu - uy^2 + ux^2 + uy^2}$$

$$= \frac{-x dx + y dy + u dy}{0}$$

$$\int -x dx + \int y dy + \int u dy = 0$$

$$= \frac{x^2}{2} + \frac{y^2}{2} + \frac{u^2}{2} = C_1 = \phi(x, y, u)$$

for $\psi(x, y, u)$

$$\frac{y dx}{yu(x+y)} = \frac{x dy}{xu(x-y)} = \frac{-u du}{-u(x^2+y^2)}$$

$$= \frac{y dx + x dy - u du}{0}$$

$$= y dx + x dy - u du = 0$$

$$\int d(xy) - \int u du = 0$$

$$xy - \frac{u^2}{2} = C_2$$

$$2xy - u^2 = \psi(x, y, u)$$

$$f(\phi, \psi) = 0$$

$$-x^2 + y^2 + x^2 + 2xy - u^2 = 0$$

$$-x^2 + y^2 + 2xy = 0$$